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The Statistics of Derangement—A Survey

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Given a finite set X of elements, divided into disjoint subsets, we define a derangement of X as a permutation which leaves none of the elements in their original subsets. The probability of a random permutation being a derangement is discussed, particularly its asymptotic value as the cardinality of X and the number of subsets tend, under certain conditions, to infinity. Finally, the problem is extended to studying the number of elements which are transferred by a general permutation to a subset other than their initial one.

KEY WORDS: Combinatorial analysis; derangements; Laguerre polynomials.

Suppose that $S_1, S_2,..., S_k$ are mutually exclusive finite sets of elements and $X = \bigcup_{i=1}^{k} S_i$. A *derangement* of X is a permutation of its elements in which no element is left in its original set. If n_i denotes the number of elements in S_i, $(1 \leqslant i \leqslant k)$, we define $D(n_1, n_2,..., n_k)$ as the total number of possible derangements of X. The probability that a random permutation is in fact a derangement is then given by

$$P(n_1, n_2,..., n_k) = D(n_1, n_2,..., n_k) \bigg/ \left(\sum_i n_i\right)!$$

The starting point for the derivation of most of the results given below is the formula[1,3]

$$D(n_1, n_2,..., n_k) = (-1)^{\sum n_i} \int_0^\infty \left\{ \prod_{i=1}^{k} L_{n_i}(x) \right\} e^{-x} \, dx \tag{1}$$

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where $L_n(x)$ is the Laguerre polynomial whose expression is

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{x^r}{r!}$$

It follows that

$$P(n_1, n_2, \ldots, n_k) = \left[(-1)^{\sum_{i=1}^{k} n_i} \Big/ \left(\sum_{i=1}^{k} n_i \right)! \right] \int_0^\infty \left\{ \prod_{i=1}^{k} L_{n_i}(x) \right\} e^{-x}\, dx \quad (2)$$

In the special case $n_1 = n_2 = \cdots - n_k = n$, it will be more convenient to use the simpler notation $P_k(n)$, $D_k(n)$. The explicit formula (2) can be exploited to prove for the function $P(n_1, n_2, \ldots, n_k)$ a variety of inequalities, recurrence relations, and asymptotic estimates, not all of which seem to be derivable directly from combinatorial considerations.

The trivial case of $P_k(1)$ is traditionally modeled by the problem of the secretary who typed k letters, with an appropriate envelope for each, but then absentmindedly put letters into envelopes simply at random. The probability that none of the letters reaches its proper destination is given by $P_k(1)$. It is easily proved by an inclusion-exclusion argument that

$$P_k(1) = \sum_{r=0}^{k} (-1)^r / r! = e^{-1} + O\left(\frac{1}{(k+1)!} \right) \quad (3)$$

This result can also be obtained directly from (2) by noting that $L_1(x) = 1 - x$.

A less trivial problem arises if each of the k letters consists of n pages and the secretary now shuffles the entire pack of nk pages before putting a random n pages into each of the k envelopes. What is now the probability that not a simple page is correctly addressed? From (2) we have

$$P_k(n) = \frac{(-1)^{nk}}{(nk)!} \int_0^\infty \{L_n(x)\}^k e^{-x}\, dx \quad (4)$$

For large k, the integral can be estimated by the Laplace method and we obtain

$$P_k(n) = e^{-n} + O(k^{-1}) \quad (5)$$

It follows incidentally, from (3) and (5), that for large k,

$$P_k(n) \sim \{P_k(1)\}^n + O(1/k) \quad (6)$$

This estimate has an obvious combinatorial interpretation, but the direct combinatorial proof is far from easy. We now come to a further generalization. Suppose that there are sets of r different sizes, $n_1, n_2,..., n_r$, (say) and k sets of each size. The probability of a derangement now becomes

$$P_k(n_1, n_2,..., n_r) = (-1)^{r\sum n_i} \int_0^\infty \left\{ \prod_{i=1}^{r} L_{n_i}(x) \right\}^k e^{-x}\, dx \Big/ \left(r \sum n_i \right)! \qquad (7)$$

This, too, can be estimated asymptotically for large k, and leads to[4]

$$P_k(n_1, n_2,..., n_r) = \exp\left\{ -\sum_{i=1}^{r} n_1^2 \Big/ \sum_{i=1}^{r} n_i \right\} + O\left(\frac{1}{k}\right) \qquad (8)$$

So far no combinatorial derivation has been found for this rather surprising result. However, it may be noted that (8) implies the combinatorial result

$$\{P_k(n_1, n_2,..., n_r)\}^{\sum_{i=1}^{n_i}} \sim \prod_{i=1}^{r} \{P_k(n_i)\}^{n_i} \qquad (9)$$

We conclude with a rather different problem. Again consider k sets $S_1, S_2,..., S_k$ of respective cardinalities $n_1, n_2,..., n_k$, where $X = \bigcup_{r=1}^{k} S_r$.

For any permutation π of X we define its shift number $\nu(\pi)$ as the number of elements of X which are moved by π to a set different from that in which they started. A derangement may thus be defined as a permutation π for which $\nu(\pi) = \sum_{r=1}^{k} n_r$.

We define the function

$$\Delta(n_1, n_2,..., n_k) = \sum_{\pi} (-1)^{\nu(\pi)} \qquad (10)$$

where the sum is taken over all permutations of X. It can be deduced from (1), using analytic properties of the Laguerre polynomials, that

$$\Delta(n_1, n_2,..., n_k) = 2^{\sum n_r + 1} \int_0^\infty e^{-2x} \left\{ \prod_{r=1}^{k} Ln_r(x) \right\} dx \qquad (11)$$

On the other hand it has been shown in ref. 2 that, if $k \leqslant 4$, the integral in (11) is always positive. We thus have an analytic proof of the combinatorial result that, for $k \leqslant 4$,

$$\Delta(n_1, n_2,..., n_k) > 0 \qquad (12)$$

Note incidentally that the condition $k \leqslant 4$ is necessary, since already

$$\Delta(1, 1, 1, 1, 1) = -8 < 0 \tag{13}$$

To close this rather curious circle, we mention that the inequality (12) can be proved by purely combinatorial methods,[5] thus providing a combinatorial proof of the positivity of the integral in (11).

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